On Exact Controllability and Complete Stabilizability for Linear Systems in Hilbert Spaces

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Abstract

A criterion of exact controllabilty using the resolvent of the state space operator is given for linear control system in Hilbert space . Only surjectivity of the semi-group operators is assumed. This condition is necessary for exact controllability, so the criterion is quite general. Relations between exact controllability and complete stabilizability are specified.

Keywords: infinite dimensional systems, linear systems, exact controllability, stabilizability

1 Introduction

We are concerned with systems described by equation

$$\dot{x} = Ax + Bu, \tag{1}$$

where x and u lie in Hilbert spaces X and U respectively. A and B are linear operators. B is bounded and A is the infinitesimal generator of a C_0-semi-group of bounded operators $S(t),\ t \geq 0$. The function u is square integrable in the sense of Bochner. The mild solution of the system (1) is given by

$$x(t, x_0, u) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau. \tag{2}$$

Definition 1.1 *The system (1) is said to be* exactly controllable *if there exists a time T such that for all $x_0, x_1 \in X$ and for some control $u(t)$, we have $x(T) = x(T, x_0, u) = x_1$.*

It is well-known (see [1, 2, 7]) that a necessary and sufficient condition of exact controllability is given by:

$$\int_0^T \|B^*S^*(\tau)x\|^2 d\tau \geq \delta_T \|x\|^2 \tag{3}$$

for some $\delta_T > 0$ and for all $x \in X$. This means that the operator K_T defined by

$$K_T x = \int_0^T S(\tau)BB^*S^*(\tau)x d\tau \tag{4}$$

is a uniformly positive definite operator and then invertible, i.e. K_T^{-1} is defined on X and bounded.
If the operator A is bounded, then this condition is equivalent to [4]:

$$\exists\, k \in \mathbb{N} \quad \text{such that} \quad \mathrm{Im}\,[\,B \quad AB \quad \dots \quad A^k B\,] = \mathrm{X}. \tag{5}$$

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This condition was used in [11] for a construction of the steering control functions which differs from the construction of [4].
When the operator A is unbounded, the situation is much more complicated. Korobov and Sklyar [3] gave a generalization of the criterion (5) for the case of an unbounded operator A which is the generator of a group.
In this paper (Section 2) we extend this result for a semi-group of surjective operators. The surjectivity of the operators $S(t), t \geq 0$ is a necessary condition for exact controllability [5, 8].
In Section 3 we consider the relation between the exact controllability and complete stabilizability, i.e. exponential stabilizability with arbitrary decay rate. We give an extension of Zabczyk's result on the relation between exact controllabilty and complete stabilizability [12] (see also [13], p. 229).

Let $\omega_0(A)$ stands for the scalar given by

$$\omega_0(A) = \lim_{t \to 0} \frac{\ln \|S(t)\|}{t}.$$

Then for all $\omega > \omega_0(A)$ there exists M_ω such that $\|S(t)\| \leq M_\omega e^{\omega t}$. If $\omega_0(A) = -\infty$, then $\omega \in \mathbb{R}$ may be choosen arbitrarily.

2 Exact Controllability

Note that the criterion (5) gives exact controllability for arbitrary time T. This condition is too strong for the case of unbounded operator. The criterion (4) depends explicitly on time T which is a priori unknown. However, from this criterion, we can give a necessary and sufficient condition where the time T does not appear.

In all the paper, the real scalar λ is assumed to be positive and $\lambda > \omega_0(A)$.

The operator $K(\lambda)$ given by

$$K(\lambda)x = \int_0^\infty e^{-2\lambda t} S(t)BB^* S^*(t)x dt, \quad x \in X,$$

is well defined and is called the extended controllability gramian (see [2]).

Proposition 2.1 *The system (1) is exactly controllable if and only if the operator $K(\lambda)$ is invertible.*

Proof. Note that the exact controllability of the system (1) is equivalent to the exact controllability of the system (see for instance [2]):

$$\dot{x} = (A - \lambda I)x + Bu, \tag{6}$$

which is characterized by the condition:

$$\int_0^T \|e^{-\lambda t} B^* S^*(t)x\|^2 dt \geq e^{-\lambda T} \delta_T(\lambda) \|x\|^2. \tag{7}$$

Suppose that $K(\lambda)$ is invertible. As it is a non negative operator then, for some $\delta(\lambda) > 0$, $\langle K(\lambda)x, x \rangle \geq \delta(\lambda)\|x\|^2$, where $\langle ., . \rangle$ denote the inner product in X. Then

$$\int_0^T \|e^{-\lambda t} B^* S^*(\tau)x\|^2 d\tau \geq \delta \|x\|^2 - \int_T^\infty e^{-2\lambda t} \|B^* S^*(t)x\|^2 dt.$$

In the other hand, for ω such that $\lambda > \omega > \omega_0(A)$, one has

$$\int_T^\infty e^{-2\lambda t} \|B^* S^*(t)x\|^2 dt \leq \frac{\|B\|^2 M_\omega^2}{2(\lambda - \omega)} e^{-2(\lambda-\omega)T} \|x\|^2.$$

Then for some $\delta_T(\lambda) > 0$ we have (7).
Conversely, if the system (1) is exactly controllable, then for some $T > 0$, the operator:

$$K_T(\lambda)x = \int_0^T e^{-2\lambda t} S(t)BB^* S^*(t)x dt, \quad x \in X,$$

is a uniformly positive definite operator and then the same holds for the operator $K(\lambda)$ since $\langle K(\lambda)x, x\rangle \geq \langle K_T(\lambda)x, x\rangle$. Hence $K(\lambda)$ is invertible. ∎

The property of exact controllability means there exists a time T for which each state x_1 is reachable from each state x_0. We may also consider the exact controllability when the time $T(x_0, x_1)$ depends on x_0 and x_1. However, as it was pointed out by Rolewicz (see [9]), there exists a universal time T of exact controllability.
From the criterion of exact controllability, one can also show that a necessary condition for exact controllability is that the operators $S(t)$ are onto (see also [5, 8]). Indeed, we have

$$K(\lambda) = \int_0^{\epsilon} e^{-2\lambda t} S(t)BB^*S^*(t)dt + \int_{\epsilon}^{\infty} e^{-2\lambda t} S(t)BB^*S^*(t)dt.$$

For exact controllability the last operator, say $K_\epsilon(\lambda)$, must be onto for some $\epsilon > 0$. But

$$K_\epsilon(\lambda) = S(\epsilon)\int_0^{\infty} e^{-2\lambda(t+\epsilon)} S(t)BB^*S^*(t+\epsilon)dt$$

and this means that $S(\epsilon)$ must be surjective. The surjectivity of $S(\epsilon)$ implies the surjectivity of $S(t)$ for all t (see [8]).
The following result is a consequence of Lemma 4.1.24 in [2] (a similar result may be found in [10] for the case of a group $S(t)$).

Proposition 2.2 *For all $x \in \mathcal{D}(A^*)$ we have $K(\lambda)x \in \mathcal{D}(A)$ and*

$$(A - \lambda I)K(\lambda)x + K(\lambda)(A^* - \lambda I)x = -BB^*x. \tag{8}$$

Let $R_\lambda = (A - \lambda I)^{-1}$ be the resolvent of A and $T_\lambda = AR_\lambda = I + \lambda R_\lambda$. As in [3] we have the following statement.

Corollary 2.3 *The operator $K(\lambda)$ may be written as*

$$K(\lambda) = 2\lambda R_{2\lambda}BB^*R_{2\lambda}^* + T_{2\lambda}K(\lambda)T_{2\lambda}^*. \tag{9}$$

Proof. From (8) we get $(A - 2\lambda I)K(\lambda)x + K(\lambda)(A^* - 2\lambda I)x + 2\lambda K(\lambda)x = -BB^*x$ for all $x \in \mathcal{D}(A^*)$. Then for all $y \in X$, we have $R_{2\lambda}^* y \in \mathcal{D}(A^*)$ and

$$(A - 2\lambda I)K(\lambda)R_{2\lambda}^* y + K(\lambda)y + 2\lambda K(\lambda)R_{2\lambda}^* y = -BB^*R_{2\lambda}^* y.$$

This gives the operator equality:

$$2\lambda K(\lambda)R_{2\lambda}^* + 2\lambda R_{2\lambda}K(\lambda) + 4\lambda^2 R_{2\lambda}K(\lambda)R_{2\lambda}^* = -2\lambda R_{2\lambda}BB^*R_{2\lambda}^*.$$

Replacing $2\lambda R_{2\lambda}$ by $T_{2\lambda} - I$ in the left hand side of this equality then completes the proof. ∎

From the Corollary 2.3 we can obtain the following expansion for $K(\lambda)$:

$$K(\lambda) = 2\lambda \sum_{k=0}^{n-1} R_{2\lambda}T_{2\lambda}^k BB^* T_{2\lambda}^{*k} R_{2\lambda}^* + T_{2\lambda}^n K(\lambda) T_{2\lambda}^{*n}. \tag{10}$$

Lemma 2.4 *Suppose that $S(t)$ $t \geq 0$, are surjective, then for all $x \in X$ we have*

$$\lim_{n\to\infty} T_{2\lambda}^{*n} x = 0$$

and then

$$K(\lambda)x = 2\lambda \sum_{k=0}^{\infty} R_{2\lambda}T_{2\lambda}^k BB^* T_{2\lambda}^{*k} R_{2\lambda}^* x. \tag{11}$$

Proof. The proof is similar to the proof given in [3, 10] for the operator $A(A+\lambda I)^{-1}$, where A is the infinitesimal generator of a group of linear operators. Here we use the equivalent norm:

$$\|x\|_0 = \sqrt{\langle K_0(\lambda)x, x\rangle}, \quad K_0(\lambda) = \int_0^\infty e^{-2\lambda t} S(t)S^*(t)dt.$$

The operator $K_0(\lambda)$ being uniformly positive definite since $S(t)$ is assumed to be surjective.

From (9), with $B = I$, we have

$$\langle K_0(\lambda)x, x\rangle = \langle 2\lambda R_{2\lambda} R_{2\lambda}^* x, x\rangle + \langle T_{2\lambda} K(\lambda) T_{2\lambda}^* x, x\rangle = 2\lambda \|R_{2\lambda}^* x\|^2 + \|T_{2\lambda}^*\|_0^2. \tag{12}$$

Then, for the given norm, $\|T_{2\lambda}x\|_0 < \|x\|_0$, for $x \neq 0$, i.e. $T_{2\lambda}$ is a *completely non-unitary* contraction (see [6]). In the other hand, a direct computation yields that the spectrum of $T_{2\lambda}$ is given by

$$\sigma(T_{2\lambda}) = \{1\} \cup \{\frac{\mu}{\mu - 2\lambda}, \ \mu \in \sigma(A)\}.$$

As $\lambda > \omega_0 \geq \Re\,\mu$, where $\Re\,\mu$ is the real part of μ, one can easely verify that $\sigma_1 = \sigma(T_{2\lambda}) \cap \{\alpha \in \mathbb{C} : |\alpha| = 1\} = \{1\}$. The measure of σ_1 is 0. Then, by a theorem of Cz.-Nagy and Foiaş (see [6], Proposition II. 6. 7) we have:

$$\forall x \in X, \quad \lim_{n\to\infty} T_{2\lambda}^{*n} x = 0, \quad \lim_{n\to\infty} T_{2\lambda}^n x = 0.$$

From this and (10) we get (11). ∎

Let $l^2(U)$ be the Hilbert space of all sequences $\{u_k, k = 0, 1, \ldots\}$, with $u_k \in U$ such that $\sum_{k=0}^\infty \|u_k\|^2 < \infty$. Consider the operator $C_\lambda(A, B)$ defined by

$$C_\lambda(A, B)w = \sum_{k=0}^\infty R_{2\lambda} T_{2\lambda}^k B u_k, \quad w = \{u_k, k = 0, 1, \ldots\},$$

It is easy to see that $C_\lambda(A, B)$ is bounded and, from (11), that $K(\lambda) = 2\lambda C_\lambda(A, B) C_\lambda^*(A, B)$. This yields the following statement.

Theorem 2.5 *The system (1) is exactly controllable if and only if $C_\lambda(A, B)$ is surjective, i.e. iff, for all $x \in X$ there exists a square summable sequence $\{u_i, i = 0, 1, \ldots\}$ such that*

$$R_{2\lambda} B u_0 + R_{2\lambda} T_{2\lambda} B u_1 + \ldots + R_{2\lambda} T_\lambda^k B u_k + \ldots = x.$$

Proof. The operator $C_\lambda(A, B)$ is surjective if and only if for some constant c and for all $x \in X$, we have $\|C_\lambda^*(A, B)x\|^2 \geq c\|x\|^2$. This equivalent to

$$(K(\lambda)x, x) \geq 2\lambda c \|x\|^2,$$

which means that $K(\lambda)$ is uniformly positive definite and therefore invertible. ∎

For the case of a bounded operator, the theorem gives the criterion (5). The proof is given in [3, 10].

3 Complete Stabilizability

We first give a precise definition of the complete stabilizability.

Definition 3.1 *The system (1) is said to be* completely stabilizable *if for all $\omega \in \mathbb{R}$ there exists a linear bounded operator $F : X \to U$ and a constant $M > 0$ such that the semi-group generated by $A + BF$, say $S_F(t)$, verifies:*

$$\|S_F(t)\| \leq M e^{\omega t} \quad \text{for} \quad \text{t} \geq 0.$$

Exact controllability implies complete stabilizability (see [13]). The converse was established ([12, 13]) for a group. Our result holds for surjective semi-group, with a minimal assumption.

If $S(t)$ are surjective, then for all $t \geq 0$, for $\delta_t = \inf\{\sigma(S(t)S^*(t)),\ t \geq 0\}$ and for all $x \in X$, we have:

$$\|S^*(t)x\|^2 \geq \delta_t \|x\|^2.$$

We make the following assumption.
Assumption A: There exists $\alpha > -\infty$ such that

$$\inf\{\frac{\ln \delta_t}{t},\ t > 0\} = \alpha.$$

Theorem 3.2 *If the system (1) is completely stabilizable and $S(t)$ is a semi-group of surjective operators satisfying Assumption A, then the system is exactly controllable.*

Proof. Suppose that the system is completely stabilizable. Then for arbitrary $\omega \in \mathbb{R}$ there exists $M > 0$ and F such that

$$\|S_F^*(t)x\| \leq Me^{\omega t}$$

for all $x, \|x\| = 1$. The semi-group $S_F^*(t)$ may be expressed by (see for example [1]):

$$S_F^*(t)x = S^*(t)x + \int_0^t S^*(t-\tau)F^*B^*S^*(\tau)x d\tau, \quad x \in X.$$

This gives

$$\|S^*(t)x\| - \int_0^t \|S^*(t-\tau)F^*B^*S^*(\tau)x\| d\tau \leq \|S_F^*(t)x\| \leq Me^{\omega t}.$$

And then

$$\|S^*(t)x\| - \left(\int_0^t \|S^*(t-\tau)F^*\|^2 d\tau\right)^{1/2} \left(\int_0^t \|B^*S^*(\tau)x\|^2 d\tau\right)^{1/2} \leq \|S_F^*(t)x\| \leq Me^{\omega t}.$$

Assume by contradiction that the system is not exactly controllable, then for all $t > 0$, for all $c > 0$, there exists $x, \|x\| = 1$, such that

$$\int_0^t \|B^*S^*(\tau)x\|^2 d\tau < c.$$

Hence

$$\|S^*(t)x\| \leq Me^{\omega t}.$$

Since $S(t)$ is surjective and by Assumption A, we get

$$e^{\frac{\alpha}{2}t} \leq \sqrt{\delta_t} \leq \|S^*(t)x\| \leq Me^{\omega t},$$

which is impossible since $\omega \in \mathbb{R}$ is arbitrary. This complete the proof. ∎

Remark The Assumption A is not very restrictive. However, it is not clear if this condition is necessary.

4 Conclusion

The result given in this paper are quite general in the case of a bounded control operator B. The case of an unbounded control operator, which include some systems described by partial differential equation with boundary control, is now under study.

References

[1] Curtain R. F., Pritchard A.J., *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, Lecture Notes in Control and Information Sciences, New York, 1978.

[2] Curtain R. F., Zwart H. J., *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[3] Korobov V. I., Sklyar G. M., *The Exact Controllability Problem for Linear System with the Operator Generating Strongly Continuous Group in Hilbert Space,* Submitted.

[4] Korobov V. I., Rabah R., *Exact Controllability in Banach Space*, Differentsialnye Uravnenia, Vol. 15, pp. 2142-2150, 1979 (in Russian).

[5] Louis J. C., Wexler D., *On Exact Controllability in Hilbert Spaces*, J. Diff. Eqns., Vol. 49, pp. 258-269, 1983.

[6] Cz.-Nagy B., Foiaş C., *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, 1967.

[7] Prikhodko A. P., Rabah R., *Some Properties for Linear Controlled Equations in Infinite Dimensional Spaces*, Vestnik Kharkov. Universiteta, PMM, Vyp. 43, pp. 37-43, 1978 (in Russian).

[8] Rabah R., *Commandabilité des Systèmes Linéaires à Retard Constant dans les Espaces de Banach*, RAIRO-APII (Automatic Control and Production Systems), Vol. 20, No 6, pp. 529-540, 1986.

[9] Rolewicz S., *On Universal time for Controllability of time-depending Linear Control Systems,* Studia Math., Vol. 59, No 2, pp. 133-138, 1976.

[10] Sklyar G. M., *On exact Controllability for Differential Equation with Unbounded Operator*, Vestnik Kharkov. Universiteta, PMM, No 361, pp. 20-27, 1991 (in Russian).

[11] Triggiani R., *Constructive Steering Control Functions for Linear Systems and Abstract Rank Conditions,* J. Optimization Theory and Applications, Vol. 74, No 2, 1992, pp. 347-367.

[12] Zabczyk J., *Complete Stabilizability Implies Exact Controllability,* Seminarul Ecuati Functionale, University of Timisoara, No 38, pp. 1-7, 1976.

[13] Zabczyk J., *Mathematical Control Theory: An Introduction*, Birkhaüser Ed., 1992.